

Convergence properties of $\eta \rightarrow 3\pi$ in low energy QCD

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Abstract

Even today, the convergence of the decay widths and some of the Dalitz plot parameters of the $\eta \rightarrow 3\pi$ decays seems problematic in low energy QCD. We provide an overview of the current experimental and theoretical situation with historical background and summarize our recent results, which explore the question of compatibility of experimental data with a reasonable convergence of a carefully defined chiral series in the framework of resummed chiral perturbation theory.

1. Overview

From the very beginning it was understood that the $\eta \rightarrow 3\pi$ decays are isospin breaking processes. Initially, the decays were thought to be of electromagnetic origin [1, 2], generated by the isospin breaking virtual photon exchange

$$H_{QED}(x) = -\frac{e^2}{2} \int dy D^{\mu\nu}(x-y) T(j_\mu(x) j_\nu(y)). \quad (1)$$

Simultaneously, however, it was discovered that the decays are almost forbidden in the framework of QED (the Sutherland theorem [3, 4]), which was met with some disbelief [2].

The early calculations [1, 2], applying current algebra and PCAC, related the $\eta\text{-}\pi$ matrix elements to the difference of squared kaon masses or kaon and pion masses, respectively. In fact, the latter resembles the later Dashen's theorem, which cannot be justified by electrodynamics [4]. In spite of that, the obtained value for the neutral channel decay rate were of approximately correct order of magnitude, $\Gamma^0=160$ eV.

Hence it became apparent that there has to be a source of isospin breaking beyond the term (1) [5]. As is known today, strong interactions break isospin via the difference between the masses of the u and d quarks

$$H_{QCD}^{IB}(x) = \frac{m_d - m_u}{2} (\bar{d}(x)d(x) - \bar{u}(x)u(x)). \quad (2)$$

The work [5] collected all the relevant current algebra terms contributing to the decays and can be considered to be the first to provide the correct leading order calculation. The obtained value for the neutral decay rate did not significantly change though and turned out to be much lower than the experimental value then available ($\Gamma^0=164$ eV vs 750 ± 200 eV). There remained a significant discrepancy, as was concluded in [5].

When a systematic approach to low energy hadron physics was born in the form of chiral perturbation theory (χ PT) [6, 7, 8], it was quickly applied to the $\eta \rightarrow 3\pi$ decays [9]. The one loop corrections were very sizable, the result for the decay width of the charged channel was $\Gamma^+=160 \pm 50$ eV, compared to the current algebra prediction of 66 eV. However, already at that time there were hints that the experimental value is still much larger (340 ± 100 eV), thus “resurrecting the puzzle” the theory aimed to solve. The current PDG value [10] is

$$\Gamma_{\text{exp}}^+ = 300 \pm 12 \text{ eV}. \quad (3)$$

In the case of the neutral channel, the average is [10]

$$\Gamma_{\text{exp}}^0 = 428 \pm 17 \text{ eV}. \quad (4)$$

After the effective theory was extended to include virtual photon exchange generated by (1) [11], it was shown that the next-to-leading electromagnetic corrections to the Sutherland's theorem are very small as well [12, 13]. The theory thus seems to converge really

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$\eta \rightarrow \pi^+ \pi^- \pi^0$	a	b	d
Cr.Barrel '98 [23]	-1.22 ± 0.07	0.22 ± 0.11	0.06 (input)
KLOE '08 [24]	-1.090 ± 0.020	0.124 ± 0.012	0.057 ± 0.017
WASA '14 [27]	-1.144 ± 0.018	0.219 ± 0.042	0.086 ± 0.025
BESIII '15 [26]	-1.128 ± 0.017	0.153 ± 0.017	0.085 ± 0.018
KLOE '16 [25]	-1.095 ± 0.004	0.145 ± 0.006	0.081 ± 0.007
NREFT '11 [22]	-1.213 ± 0.014	0.308 ± 0.023	0.050 ± 0.003
NNLO χ PT '07 [14]	-1.271 ± 0.075	0.394 ± 0.102	0.055 ± 0.057

Table 1: Recent experimental and theoretical results for $\eta \rightarrow \pi^+ \pi^- \pi^0$.

slowly for the decays. At last, the two loop χ PT calculation [14] has succeeded to provide a reasonable prediction for the decay widths.

Meanwhile, experimental data are being gathered with increasing precision in order to make more detailed analysis of the Dalitz plot distribution possible. Comparison of the recent experimental information with the NNLO χ PT results can be found in tables 1 and 2, with the conventionally defined Dalitz plot parameters defined as

$$\eta \rightarrow \pi^0 \pi^+ \pi^- : |A|^2 = A_0^2(1 + ay + by^2 + dx^2 + \dots) \quad (5)$$

$$\eta \rightarrow 3\pi^0 : |\bar{A}|^2 = \bar{A}_0^2(1 + \alpha z + \dots), \quad (6)$$

where $x \sim u-t$, $y \sim s_0-s$, $z \sim x^2+y^2$ and s_0 is the Dalitz plot center $s_0 = 1/3(M_\eta^2 + 3M_\pi^2)$. For the sake of brevity, we added the systematic and statistical uncertainties in squares. As can be seen, a tension between χ PT and experiments appears to be in the charged decay parameter b and the neutral decay parameter α .

Alternative approaches were developed in order to model the amplitudes more precisely, namely dispersive approaches [15, 16, 17, 18, 19] and non-relativistic effective field theory [20, 21, 22]. These more or less abandon strict equivalence to χ PT and succeed in reproducing a negative sign for α (see table 2)

Thence comes our motivation to ask whether it is possible to carefully define an amplitude with reasonable convergence properties which would reproduce the experimental data for the decay widths and the Dalitz plot parameters. In other words, we aim to investigate the question of compatibility of the experimental data with a reasonable convergence of the chiral series.

2. Calculation

There is a long standing suspicion that chiral perturbation theory might possess slow or irregular con-

$\eta \rightarrow \pi^0 \pi^0 \pi^0$	α
Crystal Barrel '98 [28]	-0.052 ± 0.020
SND '01 [29]	-0.010 ± 0.023
Crystal Ball '01 [30]	-0.031 ± 0.004
CELSIUS/WASA '07 [31]	-0.026 ± 0.014
WASA at COSY '09 [32]	-0.027 ± 0.009
Crystal Ball at MAMI-B '09 [33]	-0.032 ± 0.003
Crystal Ball at MAMI-C '09 [34]	-0.0322 ± 0.0025
KLOE '10 [35]	-0.0301 ± 0.0050
PDG '14 [10]	-0.0315 ± 0.0015
NREFT '11 [22]	-0.0246 ± 0.0049
Prague disp.fit '11 [18]	-0.044 ± 0.004
Bern disp.fit '11 [17]	-0.045 ± 0.010
Guo et al. '15 [19]	-0.022 ± 0.004
NNLO χ PT '07 [14]	$+0.013 \pm 0.032$

Table 2: Recent experimental and theoretical results for $\eta \rightarrow 3\pi^0$.

vergence in the case of the three light quark flavours [36, 37]. An alternative method, now dubbed resummed χ PT [38, 39], was developed in order to incorporate such a possibility. The starting point is the realization that the standard approach to χ PT, as a usual treatment of perturbation series, implicitly assumes good convergence properties and hides the uncertainties associated with a possible violation of this assumption. The resummed procedure uses the same standard χ PT Lagrangian and power counting, but only expansions derived linearly from the generating functional are considered safe. All subsequent manipulations are carried out in a non-perturbative algebraic way. The expansion is done explicitly to next-to-leading order and higher orders are collected in remainders. These are not neglected, but retained as sources of error, which have to be estimated.

The working hypothesis of the resummed approach is that only a limited set of safe observables, as defined above, has the property of global convergence, i.e. that the NNLO remainders are of a natural order of magnitude. Observables derived from the safe ones by means of nonlinear relations do not in general satisfy the criteria for global convergence due to the possible irregularities of the chiral series. Therefore, it is necessary to express such dangerous observables in terms of the safe ones in a non-perturbative way.

Our calculation is described in depth in [40]. What we present here is only a brief excerpt, meant as a sum-

mary of the basic steps and obtained results.

Within the formalism, we start by expressing the charged decay amplitude in terms of the 4-point Green functions G_{ijkl} . We compute at first order in isospin braking. In this case the amplitude takes the form

$$F_\pi^3 F_\eta A(s, t, u) = G_{+-83} - \varepsilon_\pi G_{+-33} + \varepsilon_\eta G_{+-88} + \Delta_{G_D}^{(6)}, \quad (7)$$

where $\Delta_{G_D}^{(6)}$ is the direct higher order remainder to the complete 4-point Green function. The physical mixing angles to all chiral orders and first in isospin braking can be expressed in terms of quadratic mixing terms of the generating functional to NLO and related indirect remainders

$$\varepsilon_{\pi,\eta} = -\frac{F_0^2}{F_{\pi^0,\eta}^2} \frac{(M_{38}^{(4)} + \Delta_{M_{38}}^{(6)}) - M_{\eta,\pi^0}^2 (Z_{38}^{(4)} + \Delta_{Z_{38}}^{(6)})}{M_\eta^2 - M_{\pi^0}^2}. \quad (8)$$

In this approximation the neutral decay channel amplitude can be related to the charged one as

$$\bar{A}(s, t, u) = A(s, t, u) + A(t, u, s) + A(u, s, t). \quad (9)$$

As dictated by the method, $\mathcal{O}(p^2)$ parameters appear inside loops, while physical quantities in outer legs. Due to the leading order masses in loops, such a strictly derived amplitude has an incorrect analytical structure, cuts and poles are placed in unphysical positions. To account for this, we carefully modify the amplitude using a NLO dispersive representation. The procedure is described in detail in [40].

The next step is the treatment of the low energy constants (LECs). The leading order ones, as well as the quark masses, are expressed in terms of convenient parameters

$$Z = \frac{F_0^2}{F_\pi^2}, \quad X = \frac{2F_0^2 B_0 \hat{m}}{F_\pi^2 M_\pi^2}, \quad r = \frac{m_s}{\hat{m}}, \quad R = \frac{(m_s - \hat{m})}{(m_d - m_u)}, \quad (10)$$

where $\hat{m} = (m_u + m_d)/2$. The standard approach tacitly assumes values of X and Z close to one, which means that the leading order terms should dominate the expansion. However, recent fits [41] indicate much lower values. A possibility of a non-standard scenario of spontaneous chiral symmetry breaking is thus still open.

At next-to-leading order, the LECs L_4 - L_8 are algebraically reparametrized in terms of pseudoscalar masses, decay constants and the free parameters X , Z and r using chiral expansions of two point Green functions, similarly to [38]. Because expansions are formally not truncated, each generates an unknown higher order remainder.

We still don't have a similar procedure for L_1 - L_3 . Therefore we collect a set of standard χ PT fits [42, 43, 44, 41] and by taking their mean and spread, while ignoring the much smaller reported error bars, we obtain an estimate of their influence. As is shown in [40], the results depend on these constants only very weakly. The error bands given below include the estimated uncertainties in L_1 - L_3 .

The $\mathcal{O}(p^6)$ and higher order LECs, notorious for their abundance, are collected in a relatively smaller number of higher order remainders. We have a direct remainder to the 4-point Green function and eight indirect ones - three related to each the pseudoscalar masses and the decay constants, two to the mixing angles. The last step leading to numerical results is their estimate. We use an approach based on general arguments about the convergence of the chiral series [38], which leads to

$$\Delta_G^{(4)} = (0 \pm 0.3) G, \quad \Delta_G^{(6)} = (0 \pm 0.1) G, \quad (11)$$

where G stands for any of our 2-point or 4-point Green functions, which generate the remainders. This is in principle an assumption. Hence we test the compatibility of this assumption of a reasonably good chiral convergence of trusted quantities with experimental data in a statistical sense.

3. Summary of results

Our results depend, besides the remainders, on several free parameters - the chiral condensate, the chiral decay constant, the strange quark mass and the difference of the light quark masses. They are expressed in terms of the parameters X , Z , r and R , respectively. The quark mass parameters have been fixed from lattice QCD averages [45]: $r = 27.5 \pm 0.4$ and $R = 35.8 \pm 2.6$.

We have treated the uncertainties in the higher order remainders and other parameters statistically and numerically generated a large range of theoretical predictions, which can be confronted with experimental information. Let us stress that at this point our goal is not to provide sharp predictions, as the theoretical uncertainties are large. Nevertheless, in this form, the approach is suitable for addressing questions which might be difficult to ask within the standard framework.

Full results can be found in [40], the main ones are reproduced here in figures 1 and 2.

In the case of the decay widths, the experimental values can be reconstructed for a reasonable range of the free parameters and thus no tension is observed, in spite of what some of the traditional calculations suggest [5, 9]. As can be seen in figures 1 and 2, we have

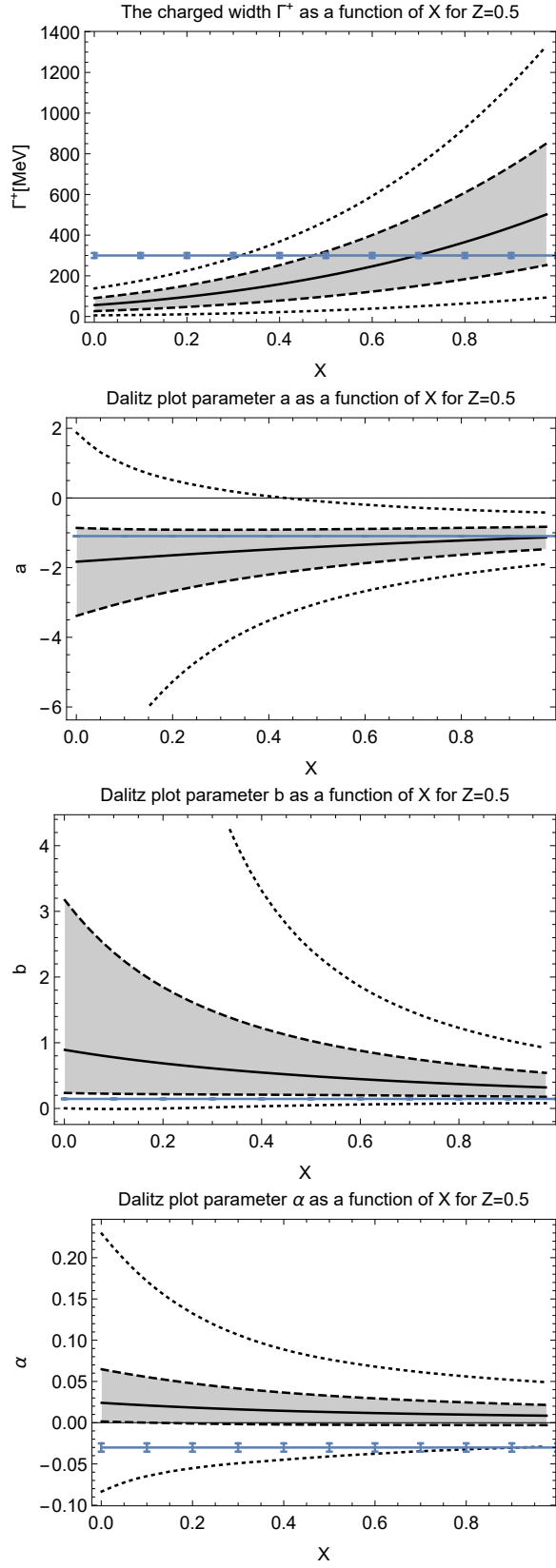


Figure 1: Parameters Γ^+ , a , b and α as a function of X for $Z = 0.5$. The median (solid line), the one-sigma band (dashed, shadowed) and the two-sigma band (dotted) are depicted along with the experimental value [10, 25, 35] (solid horizontal line with error bars).

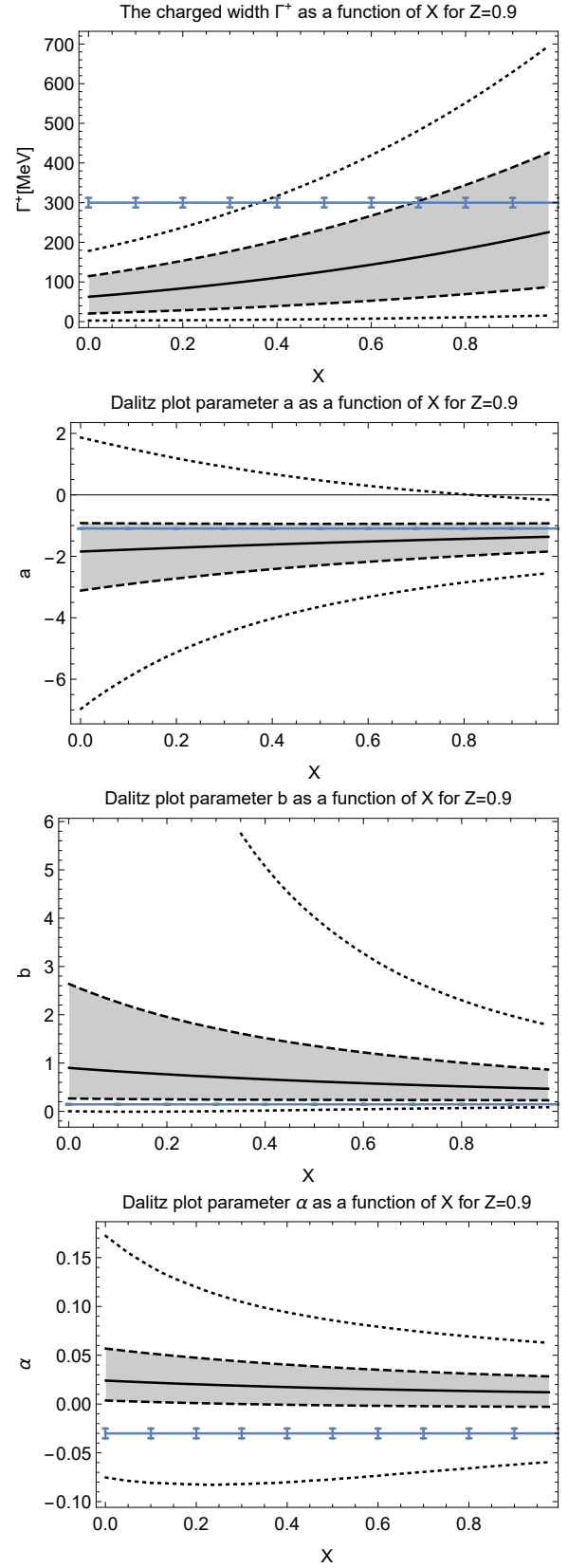


Figure 2: Parameters Γ^+ , a , b and α as a function of X for $Z = 0.9$. The median (solid line), the one-sigma band (dashed, shadowed) and the two-sigma band (dotted) are depicted along with the experimental value [10, 25, 35] (solid horizontal line with error bars).

found a strong dependence of the widths on X and Z and an appearance of both compatibility ($< 1\sigma$ C.L.) and incompatibility ($> 2\sigma$ C.L.) regions. Such a behavior is not necessarily in contradiction with the global convergence assumption and, moreover, it might be promising for constraining the parameter space and an investigation of possible scenarios of the chiral symmetry breaking [46].

As for the Dalitz plot parameters, a and d can be described very well too, within 1σ C.L. As an example, results for a are depicted in the figures.

However, when b and α are concerned, we find a mild tension for the whole range of the free parameters, at less than 2σ C.L. This marginal compatibility is not entirely unexpected. In the case of derivative parameters, obtained by expanding the amplitude in a specific kinematic point, in our case the center of the Dalitz plot, and depending on NLO quantities, the global convergence assumption is questionable, as discussed in [40]. Also, the distribution of the theoretical uncertainties is found to be significantly non-gaussian, so the consistency cannot be simply judged by the 1σ error bars.

The marginal compatibility in the case of the parameters b and α can be interpreted in two ways - either some of the higher order corrections are indeed unexpectedly large or there is a specific configuration of the remainders, which is, however, not completely improbable. This warrants a further investigation of the higher order remainders by including additional information. Work is under way in analyzing $\pi\pi$ rescattering effects and resonance contributions, some preliminary results can be found in [47].

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